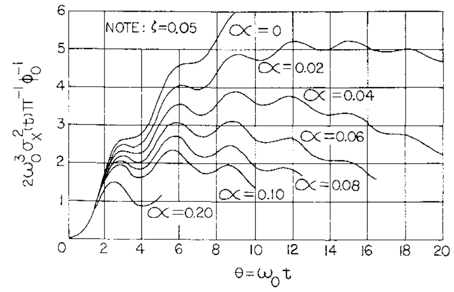

Fig. 1 Normalized variance of response to linearly decreasing random excitation.

Case 2:

$$f(t) = \exp(-\alpha\omega_0 t) \qquad 0 \leqq t \leqq T$$

$$\phi_{gg}(\omega) = \begin{bmatrix} \phi_0, & -2\omega_0 \leqq \omega \leqq 2\omega_0 \\ 0, & |\omega| > 2\omega_0 \end{bmatrix}$$

The results for Case 1 for several values of α are shown in Fig. 1 as a function of nondimensional time $\theta = \omega_0 t$; the results for Case 2 are shown in Fig. 2. For both cases, the damping was assumed to be 5% of critical. In Case 1, the excitation is suddenly applied and decreases linearly with time. In Case 2, the excitation is suddenly applied and decreases exponentially with time. For both cases, integration over η and η' in Eq. (8) has been performed analytically to obtain a closed form expression for the integrand as a function of time t and frequency ω. The remaining integral over ω was evaluated numerically to give the results shown in Figs. 1 and 2.

When $\alpha\omega_0 t = 1$ for Case 1, the excitation has become zero and from that time on the response will simply decay as a damped free oscillation. The end point on each curve in Fig. 1 represents the time at which the excitation has disappeared and represents the variance at time ($\omega_0 t = \alpha^{-1}$). The curve for $\alpha = 0$ corresponds to the response to a suddenly applied stationary excitation and is identical to results given by Caughey.[6]

Figure 2 shows that, for Case 2, the variance of the response approaches zero as time increases for all positive values of α. This occurs because the level of the excitation decreases quickly at first but then very slowly approaches zero as time increases to large values. The damping of the system dissipates energy over the entire time interval. The large amount of energy received initially is dissipated. The response then decreases toward zero along with the excitation.

When $\alpha = 0$, the equations for $\sigma_x^2(t)$ for Cases 1 and 2 become identical. Thus, the curves in Figs. 1 and 2 for $\alpha = 0$ are the same; they provide a convenient measure for the differences in $\sigma_x^2(t)$ for the two cases.

Discussion

The method used here can be applied to calculate the response of launch vehicles to nonstationary random excitation which can be written in the separated form of Eq. (1). The examples given illustrate how the method is applied to a 1-degree-of-freedom system. The numerical results show how the variance of the response changes with time. In an actual dynamic loads problem, of course, the choice of the function $f(t)$ and the power spectral density of $g(t)$ would be based on knowledge of the actual excitation of the vehicle. This knowledge would come from wind tunnel data, flight measurements, an estimate based on practical experience, or a combination of these. The method presented here can use this best estimate of the vehicle excitation to determine the nonstationary vehicle response.

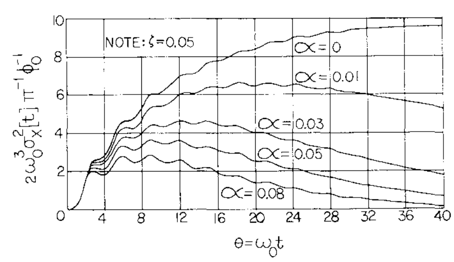

Fig. 2 Normalized variance of response to exponentially decreasing random excitation.

References

[1] Rainey, A. G., "Progress on the launch-vehicle buffeting problem," J. Spacecraft Rockets **2**, 289–299 (1965).

[2] Andrews, J. S., "The effects of transonic buffeting on a hammerhead shaped payload," *AIAA Symposium on Structural Dynamics and Aeroelasticity* (American Institute of Aeronautics and Astronautics, New York, 1965), pp. 214–226.

[3] Davis, R. E., Dick, J. W., Suhre, J. R., and Callahan, J. A., "Dynamic response of Gemini adapter module to aerodynamic excitation," *AIAA Symposium on Structural Dynamics and Aeroelasticity* (American Institute of Aeronautics and Astronautics, New York, 1965), pp. 227–238.

[4] Simon, W. E., "Flow-field parameter analysis of the wind induced oscillations of launch vehicles," *AIAA Symposium on Structural Dynamics and Aeroelasticity* (American Institute of Aeronautics and Astronautics, New York, 1965), pp. 486–505.

[5] Caughey, T. K. and Stumpf, H. J., "Transient response of a dynamic system under random excitation," J. Appl. Mech. **28**, 563–566 (1961).

[6] Caughey, T. K., "Nonstationary random inputs and responses," *Random Vibrations, Volume 2*, edited by S. H. Crandall (Massachusetts Institute of Technology Press, Cambridge, Mass., 1963), pp. 66–83.

[7] MacNeal, R., Barnoski, R. L., and Bailie, J. A., "Response of a simple oscillator to nonstationary random noise," J. Spacecraft Rockets **3**, 441–443 (1966).

Note on Coplanar Orbit Transfers by Tangential Impulses at Apse Points

Ernst D. Dickmanns*
Institut für Flugmechanik, Oberpfaffenhofen, Germany

THIS note does not contain any new results, but it is intended to give a simple and easy-to-memorize presentation of well-known solutions to some special minimum impulse transfer problems. The orbit of a body may be thought to have been generated at its apse point(s) by applying an impulse normal to the radius vector from the central mass to the body which had been at rest previously. The body at rest at a distance r_0 from the center of attraction is interpreted to be at the apocenter of a degenerate ellipse with eccentricity $e = 1$ and angular momentum $h = 0$. To discern between the apocenter and pericenter of an ellipse, the sign of the eccentricity is chosen by considering $e' = e \cos\varphi$, where φ is measured from the pericenterline and may have the values 0 and π. Now, the state of the body at rest is specified entirely by r_0, $e' = -1$, and $h = 0$.

On a plot with the square of the angular momentum h^2 and the eccentricity e as coordinates, all orbits generated by impulses normal to radius vector from the central mass are represented by points on a straight line from $e' = -1$ (Fig.

Received July 13, 1965; revision received May 23, 1966. This idea has been pursued during a NASA Fellowship at Princeton University, sponsored by the Minister of Scientific Research, Federal Republic of Germany.

* Dipl. Ing., Deutsche Versuchsanstalot für Luft-und Raumfahrt e.V.

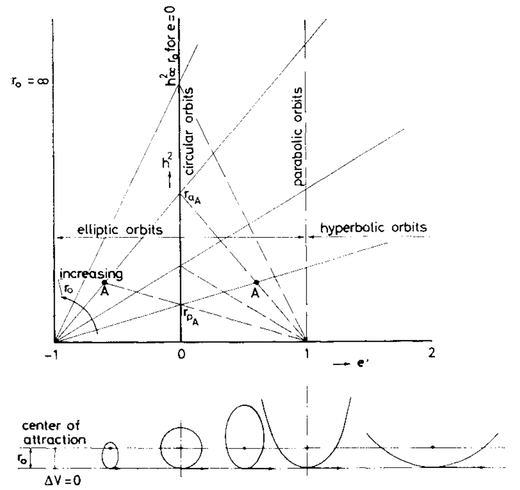

Fig. 1 Orbit generation by impulses normal to the radius vector from a central mass to a body at rest at a distance r_0.

1). The slope of this line

$$d(h^2)/de' = r_0\, GM$$

is proportional to the initial distance r_0 from the central body. For coplanar orbits, with the same center of attraction and axis orientation, the orbit is specified entirely by two quantities, e.g., angular momentum h and eccentricity e. However, in the presentation chosen, each elliptic orbit occurs twice, once in an apocenter ($e' < 0$) and once in a pericenter ($e' > 0$) point. This may be unified by mirroring the ($e' < 0$) part at the $e = 0$ axis.

Now each orbit is represented by one point, and tangential impulses at apse points show up as line segments on the rays originating from the points $h = 0$, $e' = \pm 1$. Tangential impulses at the apocenter of an elliptic orbit lie on rays from $e' = 1$ and those at the pericenter lie on rays from $e' = -1$. Instead of crossing the centerline $e = 0$, they are reflected at it, indicating that the former apocenter is now pericenter and vice versa.

This graph permits a simple presentation of coapsidal and cotangential impulsive transfers. It has been shown by Ting[1] and recently for the special case of transfers starting from a circular orbit by Moyer,[2] that this class furnishes the minimum impulse transfers between coplanar elliptic orbits with free axis-orientation.

By scaling all orbits by the initial orbit angular momentum, the initial orbit lies on the line $\hat{h}^2 = 1$. Candidates for the minimum impulse transfer from an initial orbit A to a final orbit are shown in Fig. 2. There are orbit regions in the $\hat{h}^2$, e-plane to which two impulse transfers like A-V-P or

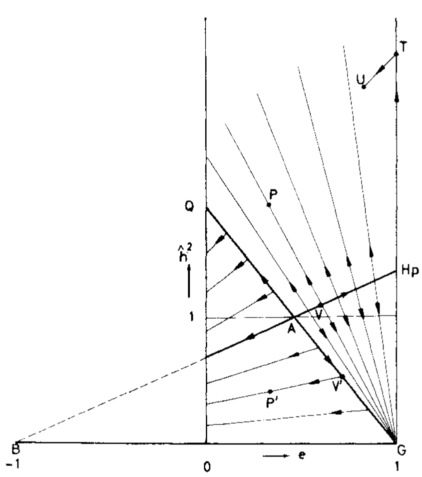

Fig. 2 Candidates for minimum impulse orbit transfers between coapsidal ellipses.

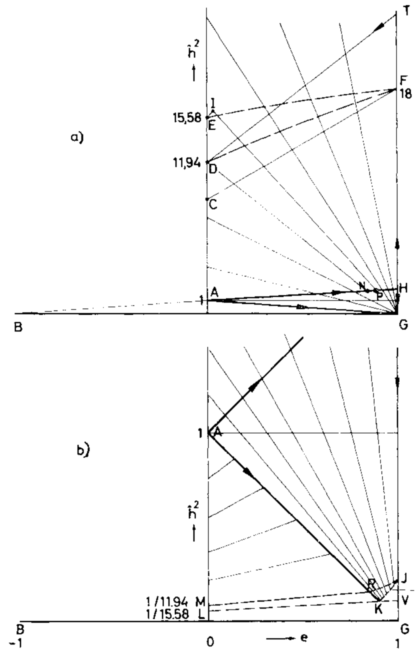

Fig. 3 Transfer from circular initial orbit: a) outbound, b) inbound.

A-V'-P' yield minimum velocity requirement (the initial orbit A lies in this region); and there are other regions to which the minimum impulse transfer goes via three impulses, one of them applied at $r = \infty$ on the parabolic orbit H_p (A-H_p-T-U).

To the author's knowledge, the boundary between these regions, depending on the eccentricity e_0 of the starting ellipse, is not known entirely. However, in Ref. 3, a procedure is devised for generating any optimum n-impulse transfer. For the special case of $e_0 = 0$, the numerical solution is given in Ref. 2 and is partly reproduced in Fig. 3 with the proposed ordinate scale. Within the region $AMRJFDA$, the two-impulse Hohmann-like transfer yields the minimum velocity requirement. Outside this region, the absolute minimum is furnished by three impulse maneuvers, in which the first impulse always transfers the vehicle to a parabolic orbit H. At $r = \infty$, an infinitesimal impulse theoretically permits the orbit to be changed to any parabolic one represented on the vertical line radiating from $e = 1$. Finally, a third impulse at the apse point of the new parabola transfers the vehicle to the target orbit.

Although the three-impulse transfer yields the absolute minimum ΔV outside the lines $\overline{DF}$ and $\overline{MRJ}$, within the regions $DEFD$ and $MRJKLM$, the two-impulse transfer still furnishes a local minimum. Target orbits represented by points above line $\overline{AH}$ lie entirely outside the original circular orbit; those below line $\overline{AG}$ are entirely inside. Final orbits in the region $AHGA$ intersect the starting orbit. For both two- and three-impulse nonintersecting inbound transfers, the final impulse is applied at the periapsis opposing the velocity vector. Therefore, the borderlines where the two-impulse transfer ceases to be globally minimizing ($\overline{MR}$), and where it stops being locally minimizing ($\overline{LK}$), have to be rays from the degenerate orbit point $e' = -1$, $h = 0$. The orbits on each of these lines all have the same pericenter distance given by the slope of these rays.

References

[1] Ting, L., "Optimum orbital transfer by impulses," ARS J. **30**, 1013–1018 (1960).

[2] Moyer, H. G., "Minimum impulse coplanar circle-ellipse transfers," AIAA J. **3**, 723–725 (1965).

[3] Breakwell, J. V., "Minimum impulse transfer," AIAA Paper 63-416 (1963).